

# SOLUTION OF WALD'S GAME USING LOADINGS AND ALLOWED STRATEGIES

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**ABSTRACT.** We propose a new interpretation of the strange phenomena that some authors have observed about the Wald game. This interpretation is possible thanks to the new language of *loadings* that Morrison and the author have introduced in a previous work. Using the theory of loadings and allowed strategies, we are also able to prove that Wald's game admits a *natural* solution and, as one can expect, the game turns out to be fair for this solution. As a technical tool, we introduce the notion of *embedding a game into another game* that could be of interest from a theoretical point of view. *En passant* we find a very easy example of a game which is loadable in infinitely many different ways.

## 1. INTRODUCTION

Consider Wald's game *pick the bigger integer*: the set of pure strategies is the set of non-negative integers and the payoff function of player 1 (which is the negative of the payoff function of players 2) is

$$f(s, t) = \begin{cases} 1, & \text{if } s > t \\ 0, & \text{if } s = t \\ -1, & \text{if } s < t \end{cases}$$

Wald[Wa] proved that this game has no value if just countably additive strategies are allowed. On the other hand, in [He-Su] it has been shown that this game has a value if one allows finitely additive probability measures as strategies, but the value depends on the order of integration in such a way that the *internal* player has an advantage<sup>1</sup>. So a game which is naturally symmetric turns out to be asymmetric.

This note has two goals: first we want to show that this apparently strange situation is perfectly explained and natural looking at the game from the point of view of the theory of loadings and allowed strategies; second we want to propose a natural solution of the game using again these new notions. It is interesting the fact that Wald's game turns out to be fair for such a *natural* solution, that is exactly what one would expect.

In the next section we recall very briefly some of the definitions given in [Ca-Mo] and we introduce the notion of embedding of games that could be of interest from a theoretical point of view (see Remark 2). The final section is devoted to the results that motivate this note.

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<sup>1</sup>See [Sc-Se] for more general results and relations with other phenomena, as de Finetti's non-conglomerability.

## 2. BASIC NOTIONS

Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two  $n$ -player games and let  $P_j^i$  (resp.  $S_j^i$ ), for  $i \in \{1, \dots, n\}$  and  $j = 1, 2$ , be the set of pure (resp. mixed) strategies of the  $i$ -th player of the  $j$ -th game. Let us denote by  $\pi_j$  the payoff function of the game  $\mathcal{G}_j$ . Given a map  $f : P_1^i \rightarrow P_2^i$  and  $\mu \in S_1^i$ , we can consider the push-forward strategy  $f_*\mu \in S_2^i$ .

**Definition 1.** *We say that*

- (1)  $\mathcal{G}_1$  embeds into  $\mathcal{G}_2$  if there exist  $n$  injective maps  $f_i : P_1^i \rightarrow P_2^i$  such that for all  $(s_1^1, \dots, s_1^n) \in S_1^1 \times \dots \times S_1^n$  one has

$$\pi_1(s_1^1, \dots, s_1^n) = \pi_2(f_{1*}s_1^1, \dots, f_{n*}s_1^n)$$

- (2)  $\mathcal{G}_1$  is isomorphic to  $\mathcal{G}_2$  if each  $f_i$  is bijective.

**Remark 2. (Approximation of games)** *The notion of embedding of games leads very naturally to some notion of approximation of a game by taking an increasing family of subgames that converges in some reasonable sense. This could be a useful tool to study complicated games using easier ones. In order to develop a good theory, one would need some theorem of convergence and this is certainly the purpose of future research. Here we are just interested in giving a first application of this notion.*

Recall that a *semigroup*  $S$  is a set equipped with an associative binary operation  $S \times S \rightarrow S$ . Given  $x, y \in S$ , the result of the operation is denoted by  $xy$ .

**Definition 3.** *Let  $S$  be a semigroup and let  $W$  be a subset of  $S$ . Fix a function  $h : S \rightarrow [-1, 1]$ . The Operation Game associated to  $S, W$  and  $h$ , denoted by  $\mathcal{G}(S, W, h)$ , is the two-person zero-sum game with  $S$  the set of pure strategies for both players: player 1 chooses  $x \in S$  and player 2 chooses  $y \in S$ , then player 1 wins if  $xy$  is in  $W$ . The payoff to player 1, which is the negative of the payoff to player 2, is  $h(xy)$ .<sup>2</sup>*

**Remark 4.** *We can also consider the more general case in which the set of pure strategies of player  $i$  is just a subset  $S_i$  of  $S$ . In this case the Operation Game will be denoted by  $\mathcal{G}(S_1, S_2, S, W, h)$ . In particular we are interested in the operation game  $\mathcal{G}(\mathbb{N}, -\mathbb{N}, \mathbb{Z}, \mathbb{N}, \chi_{\mathbb{N}} - \chi_{\mathbb{Z} \setminus \mathbb{N}})$ , so that the payoff to player 1 is*

$$f(x, y) = \begin{cases} 1, & \text{if } x + y \in \mathbb{N} \\ 0, & \text{if } x + y = 0 \\ -1, & \text{if } x + y \in -\mathbb{N} \end{cases}$$

In [Ca-Mo] it is shown that loadings play a fundamental rule to play in a coherent way and to solve the Operation Game. Indeed in general it is typical the situation that the game, which is intrinsically symmetric, loses its symmetry. Here is a sketch of the construction. First we recall the following

**Definition 5.** *A **loading** on  $S$  is any finitely additive probability measure on  $S$  which is left and right invariant with respect to the operation in  $S$ . A loading is denoted by  $\ell$ .*

The loading is fixed a priori and it is *part of the rules of the game*, in the sense that it induces a set of allowed strategies as follows

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<sup>2</sup>The definition in [Ca-Mo] was actually little different, but the reader can easily prove that they are basically equivalent. Indeed, in [Ca-Mo], we use  $h = \chi_W$  and we can pass to function taking values in  $[-1, 1]$  without modifying the game with the (usual) affine transformation  $[0, 1] \ni \lambda \rightarrow 2\lambda - 1 \in [-1, 1]$ . Finally each proof in [Ca-Mo] uses just the invariance of the measures and so it could be applied to a general function  $h$ , instead of the particular one  $\chi_W$ .

**Definition 6.** *The set of allowed strategies  $\mathcal{A}_\ell$  is any maximal set of finitely additive probability measures on  $S$  containing the loading  $\ell$  and keeping the symmetry of the game, i.e.*

$$\int_S \int_S \chi_W(xy) dp(x) dq(y) = \int_S \int_S \chi_W(xy) dq(y) dp(x)$$

for all  $p, q \in \mathcal{A}_\ell$  (where  $\chi$  stands for the characteristic function).

In [Ca-Mo] the authors have shown that allowing to play just the strategies in  $\mathcal{A}_\ell$ , then the Operation Game in the sense of Definition 3 has the value  $\ell(W)$  and there are at least one optimal strategies for both the players, which is indeed  $\ell$ . Moreover, it is shown that there are operation games which are loadable in infinitely many different ways. This basically means that such games are dramatically not well defined and it is really necessary to fix the loading a priori in order to play in a coherent way. We are going to show that this is exactly what happens in the case of Wald's game. By the way we recall that in many cases (finite games, compact games, other lucky cases) the choice of a loading is not required, because there is basically a unique way to load the game and so the game is intrinsically well defined. Unfortunately, this is not the case also for very natural groups and semigroups, such as  $(\mathbb{Z}, +)$  and  $(\mathbb{N}, \cdot)$ , as is also shown in Lemma 7.

### 3. MAIN RESULTS

The following lemma is quite unexpected but probably well-known to the experts. In our case it is necessary for the proof of the main result and also it provides a very easy example of a game which is loadable in infinitely many different ways. We recall that at the end of [Ca-Mo], it is proposed an example of a game which is loadable in infinitely many different ways, but this example is quite complicated and unnatural from the point of view of a *real* game, namely a game that can be proposed to human beings.

**Lemma 7.** *For any real number  $s \in [0, 1]$ , there is a finitely additive translation invariant probability measures  $m$  on  $\mathbb{Z}$  such that  $m(\mathbb{N}) = s$ . In particular there are uncountably many different ways to load the Operation Game  $\mathcal{G}(\mathbb{Z}, \mathbb{N})$ .*

*Proof.* Let  $k$  be a positive integer. If  $A$  is a subset of  $\mathbb{Z}$ , we define its  $k$ -density as follows

$$d_k(A) = \lim_{n \rightarrow \infty} \frac{|A \cap [-kn, n]|}{(k+1)n+1}$$

where, as usual, the notation  $[n, m]$  stands for the set of integers  $x$  such that  $n \leq x \leq m$ . Clearly there are many subsets  $A$  for which  $d_k$  does not exist, so let  $D_k = \{\chi_A, A \text{ s.t. } d_k \text{ exists}\}$  and let  $X_k \subseteq L^\infty(\mathbb{Z})$  be the linear span of  $D_k$  inside  $L^\infty(\mathbb{Z})$ . Clearly  $d_k$  extends to a linear and bounded functional on  $X_k$  which is dominated by the  $L^\infty$ -norm. Hence we can apply the Hahn-Banach extension theorem to get a positive linear functional  $\phi_k : L^\infty(\mathbb{Z}) \rightarrow \mathbb{R}$ . Now let  $\mathcal{X}_k \subseteq (L^\infty(\mathbb{Z}))^*$  be the set of all such extensions  $\phi_k$ . It is a convex and compact space with respect to the weak\*-topology. Convexity is indeed trivial and compactness follows from the fact that  $\mathcal{X}_k$  is weak\*-closed in the unit ball of  $(L^\infty(\mathbb{Z}))^*$ , which is weak\* compact by the Banach-Alouglu theorem. Now  $\mathbb{Z}$  acts on  $\mathcal{X}_k$  via translations as a commutative family of operators and then Markov-Kakutani fixed point theorem applies. Such a fixed point, say  $f_k$ , is an element in  $L^\infty(\mathbb{Z})^*$  which is fixed by every translation and such that  $f_k(1) = 1$ . It follows that setting  $m_k(A) = f_k(\chi_A)$  we get a translation invariant finitely additive probability measure on  $\mathbb{Z}$ . Now since  $f_k$  extends  $d_k$ , we have  $m_k(\mathbb{N}) = d_k(\mathbb{N}) = \frac{1}{(k+1)}$ . When  $k$  goes to infinity, this proves that there exist finitely additive

translation invariant probability measures taking over  $\mathbb{N}$  values arbitrarily close to 0. Now repeating the argument with

$$d_k(A) = \lim_{n \rightarrow \infty} \frac{|A \cap [-n, kn]|}{(k+1)n+1}$$

we also prove that there exist finitely additive translation invariant probability measures which take over  $\mathbb{N}$  values arbitrarily close to 1. Now we know - and this is basically due to Chou [Ch] - that the set of values which are taken by some finitely additive translation invariant probability measure on a fixed set  $W$  is convex and closed, so, in our case, it has to be the whole interval  $[0, 1]$ .  $\square$

**Theorem 8.** *Wald's game is equivalent to an Operation Game which is loadable in infinitely many different ways.*

*Proof.* The idea is easy: consider the Operation Game  $\mathcal{G}(\mathbb{Z}, \mathbb{N}, h)$ , with  $h(x) = \chi_{\mathbb{N}}(x) - \chi_{\mathbb{Z}-\mathbb{N}}(x)$ . We want to embed Wald's game into this game. More precisely we are going to show that Wald's game is equivalent to the game  $\mathcal{G}(\mathbb{N}, -\mathbb{N}, \mathbb{Z}, \mathbb{N}, \chi_{\mathbb{N}} - \chi_{\mathbb{Z}-\mathbb{N}})$  of Remark 4 and so in particular, the set of outcomes<sup>3</sup> is still  $\mathbb{Z}$ . This is important in order to apply Lemma 7, since it is certainly false in general that a subgame of a game which is loadable in infinitely many different ways is still loadable in infinitely many different ways, but a sufficient condition to pass this property to subgames is clearly that the set of outcomes  $S$  remains the same. Now that we have the idea, the proof is straightforward: with the notation of Definition 1, define  $f_1(x) = x$  and  $f_2(y) = -y$ . So Player 1, after transforming the game, wins if and only if  $f_1(x) + f_2(y) \in \mathbb{N}$  that happens if and only if  $\max\{x, y\} = x$ . Hence Player 1 wins in the Operation Game  $\mathcal{G}(\mathbb{N}, -\mathbb{N}, \mathbb{Z}, \mathbb{N}, \chi_{\mathbb{N}} - \chi_{\mathbb{Z}-\mathbb{N}})$  if and only if it wins in the Wald game and the payoff function is clearly preserved. This proves that these two games are isomorphic and in particular Wald's game is loadable in infinitely many different ways.  $\square$

Now we are ready to propose a solution for the Wald game. There is indeed a very natural approach to solve the game  $\mathcal{G}(\mathbb{N}, -\mathbb{N}, \mathbb{Z}, \mathbb{N}, \chi_{\mathbb{N}} - \chi_{\mathbb{Z}-\mathbb{N}})$ , given by the fact that the set of pure strategies of the first player is the opposite of the set of pure strategies of the second player. This observation leads to the fact that a loading that can be *accepted for playing by both the players* is any loading verifying  $\ell(A) = \ell(-A)$ , for all  $A \subseteq \mathbb{N}$ . It indeed reflects the point of view of both the players. It is clear that any loading verifying such equality is such that  $\ell(\mathbb{N}) = \frac{1}{2}$ . Hence

**Theorem 9.** *Wald's game can be solved in a natural way and the value of the game for this solution is 0, i.e. the game is fair.*

*Proof.* We have already observed that a natural way to solve the game consists in fixing a loading  $\ell$  verifying  $\ell(A) = \ell(-A)$ , for  $A \subseteq \mathbb{N}$ , and then playing the allowed strategies induced by  $\ell$ . The point is that we cannot use directly Definition 6 to construct the set of allowed strategies, since the mixed strategies of the players are not finitely additive probability measures on  $\mathbb{Z}$ . Indeed the strategies of Player 1 (resp. Player 2) are measure on  $\mathbb{N}$  (resp.  $-\mathbb{N}$ ). But we can take inspiration from Definition 6 and make the following construction. First of all, observe that given a measure  $\mu$  on  $\mathbb{N}$ , we can construct a probability measure  $\hat{\mu}$  on  $\mathbb{Z}$  by reflecting  $\mu$  in the following way: given  $A \subseteq \mathbb{Z}$ , we define

$$\hat{\mu}(A) = \mu(A \cap \mathbb{N}) + \mu(-((A - 1) \cap (-\mathbb{N})))$$

where  $A - 1 = \{a - 1, a \in A\}$ . Analogously we can construct a probability measure  $\hat{\mu}$  on  $\mathbb{Z}$  by reflecting a probability measure  $\mu$  on  $-\mathbb{N}$ . Let  $\mathcal{P}(X)$  be the set of probability measures on a set  $X$ ,

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<sup>3</sup>The set of outcomes in this context is the set of  $xy$ , when  $x \in S_1$  and  $y \in S_2$ .

we define the set of allowed strategies for  $\mathcal{G}(\mathbb{N}, -\mathbb{N}, \mathbb{Z}, \mathbb{N}, \chi_{\mathbb{N}} - \chi_{\mathbb{Z} \setminus \mathbb{N}})$  induced by  $\ell$  to be any subset  $\mathcal{A}_\ell \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(-\mathbb{N})$  which is maximal with respect to the following two properties:

(1)

$$\int_{\mathbb{N}} \int_{-\mathbb{N}} (\chi_{\mathbb{N}}(x+y) - \chi_{\mathbb{Z} \setminus \mathbb{N}}(x+y)) dp(x) dq(y) = \int_{-\mathbb{N}} \int_{\mathbb{N}} (\chi_{\mathbb{N}}(x+y) - \chi_{\mathbb{Z} \setminus \mathbb{N}}(x+y)) dq(y) dp(x)$$

for all  $(p, q) \in \mathcal{A}_\ell$ .

(2)  $\ell \in \{\hat{\mu}, \mu \text{ allowed strategy}\}$

Observe that this maximal set exists and contains the pair  $(\check{\ell}_1, \check{\ell}_2)$ , where  $\check{\ell}_1 \in \mathcal{P}(\mathbb{N})$  and  $\check{\ell}_2 \in \mathcal{P}(-\mathbb{N})$  are defined by setting

$$\check{\ell}_1(A) = \ell(A) + \ell(-A)$$

for all  $A \subseteq \mathbb{N}$  ( $\check{\ell}_2 \in \mathcal{P}(-\mathbb{N})$  is defined in an analogue way). It is now easy to check, being the details basically the same as the main theorem in [Ca-Mo], that  $(\check{\ell}_1, \check{\ell}_2)$  is a profile of optimal strategies and that the value of the game is indeed  $2\ell(\mathbb{N}) - 1 = 2 \cdot \frac{1}{2} - 1 = 0$ .  $\square$

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